Due Thursday, December 7, 14:00, in the mailbox in the lobby of Research I.

(1) Let \((x_1, y_1), (x_2, y_2), (x_3, y_3)\) be the vertices of a triangle \(T\) in \(\mathbb{R}^2\). Show that the area of \(T\) is given by the absolute value of \(\frac{1}{2} \left| \begin{array}{ccc} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{array} \right| \).

Use this to calculate the area of the following triangles given by their vertices:

- \(T_1\): \((-3, 2), (1, 4), (-2, -7)\);
- \(T_2\): \((1, 1), (2, -1), (4, 6)\);
- \(T_3\): \((1, 1), (1, 0), (2, 3)\).

(20 points)

(2) Consider matrices \(A \in \text{Mat}(2, \mathbb{R})\).

(a) Show that \(A^2 = I_2\) if and only if \(A = \pm I_2\) or \(P_A(\lambda) = \lambda^2 - 1\).

(b) Show that \(A^2 = 0\) implies \(P_A(\lambda) = \lambda^2\). (10+10 points)

(3) Show that \(\begin{vmatrix} x & y & 0 & 1 \\ -y & x & -1 & 0 \\ 0 & 1 & x & -y \\ -1 & 0 & y & x \end{vmatrix} = (x^2 + y^2 + 1)^2\). (20 points)

(4) For \(a_1, a_2, \ldots, a_n \in F\), the following Vandermonde determinant has a particularly simple form. Show by induction on \(n\) that

\[
\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \end{vmatrix} = \prod_{i,k=1}^{n-1} (a_i - a_k). \]

Hint: This is not that easy, but there are several ways to do it. Remember you will have to use the inductive hypothesis somewhere!

More hints in the tutorial...

(30 points)

(5) Let \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\) \(\in \mathbb{R}^2\), such that the \(x_i\) are pairwise distinct (i.e. \(x_i \neq x_j\) for \(i \neq j\)). Show that there exists exactly one polynomial \(p\) of degree less than \(n\) such that \(p(x_i) = y_i\) for all \(1 \leq i \leq n\). (10 points)

(6) Bonus Problem

(a) Let \(F\) be a field, \(x_1, \ldots, x_n, y_1, \ldots, y_n \in F\). Find a formula for \(\det(a_{ij})\), where \(a_{ij} = 1/(x_i + y_j)\) (and \(x_i + y_j \neq 0\)).

(b) Show that the inverse of \((a_{ij}) \in \text{Mat}(n, \mathbb{Q})\), where \(a_{ij} = 1/(i + j - 1)\), has integral entries. (10+10 points)