Linear Algebra I - Problem Set 3

Fall 2006

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Due Thursday, Sept. 28, 2006, 14:00, in the mailbox in the lobby of Research I.

(1) Let $V$ be a vector space with $\dim V = n$, and take $v_1, \ldots, v_n \in V$. Prove that the following statements are equivalent.

(i) $\{v_1, \ldots, v_n\}$ is a basis of $V$
(ii) $v_1, \ldots, v_n$ are linearly independent
(iii) $L(v_1, \ldots, v_n) = V$ (30 points)

(2) Let $V$ be a vector space and $v_1, \ldots, v_n \in V$. Show that $\dim L(v_1, \ldots, v_n) \leq n$. (20 points)

(3) Let $V$ be a real vector space, and $a, b, c, d \in V$. Show that the following vectors are linearly dependent:

$v_1 = 2a + 9d$
$v_2 = 5c$
$v_3 = a + b + c + d$
$v_4 = a + 2b + 3c + 4d$
$v_5 = a$

HINT. There is a very short solution. (20 points)

(4) Give a basis for each of the following $\mathbb{R}$-vector spaces.

$U_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 - x_2 + x_3 = 0\}$
$U_2 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + 2x_2 + 3x_3 = 0, x_1 + x_2 - x_4 = 0\}$

Use the definition of ‘basis’ from the lecture to show your answer is correct. (15+15 points)

(5) Bonus Problem

Define functions $f_n : \mathbb{R} \to \mathbb{R}$ for $n \in \mathbb{Z}$ as follows:

$f_0(x) = 1$, $f_n(x) = \sin(nx)$ (for $n \geq 1$), and $f_n(x) = \cos(-nx)$ (for $n \leq -1$).

Are the functions $f_n$, $n \in \mathbb{Z}$, linearly independent? Prove your answer! (20 points)