Introductory Number Theory - Problem Set 8

Spring 2008

Michael Stoll

Due Tuesday, April 22, 14:00, in Prof. Stoll’s office.

(1) Let $F \in \mathbb{Z}_p[x_1, \ldots, x_n]$ be a polynomial in $n$ variables, and let $\bar{F} \in \mathbb{F}_p[x_1, \ldots, x_n]$ be the polynomial obtained by reducing the coefficients mod $p$.

Let $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{F}_p^n$ be such that $\bar{F}(\xi) = 0$, but not all partial derivatives $\frac{\partial \bar{F}}{\partial x_i}$ vanish at $\xi$. Show that there is $a \in \mathbb{Z}_p^n$ reducing mod $p$ to $\xi$ such that $F(a) = 0$.

(20 points)

(2) For all primes $p$, show that there is a non-degenerate diagonal quadratic form $Q(x_1, x_2, x_3, x_4) = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$ such that $Q(x_1, x_2, x_3, x_4) = 0$ has no nontrivial solutions in $\mathbb{Q}_p$.

(20 points)

(3) Show that for every non-degenerate quadratic form $Q$ in five variables, there are nontrivial solutions of $Q = 0$ in $\mathbb{Q}_p$.

(a) for odd primes $p$, (b) for $p = 2$.

Hints. (1) HW (6.5). (2) Normalize to get something useful mod $p$.

(15+15 points)

(4) Show that for all $v$ and all $a \in \mathbb{Q} \setminus \{0, 1\}$, we have

$$\left( \frac{a - 1}{v} \right) = 1 \quad \text{and} \quad \left( \frac{a, 1 - a}{v} \right) = 1.$$  

(10 points)

(5) Verify the values of the 2-adic Hilbert Norm Residue Symbol (20.6 in the notes).

Hint. Try to reduce to only a few cases.

(20 points)

(6) Bonus Problem.

Let $Q$ be a non-degenerate ternary quadratic form. Show that $Q$ rationally represents every nonzero integer $n$ (i.e., there are $x, y, z \in \mathbb{Q}$ such that $Q(x, y, z) = n$) if and only if $Q(x, y, z) = 0$ has a primitive integral solution.

Does the equivalence still hold if we restrict to $n > 0$?

(20+10 points)

(7) Bonus Problem.

Define $a_1 = 1$ and $a_{n+1} = \frac{1 + a_1^2 + \cdots + a_n^2}{n}$ for $n \geq 1$.

Decide whether all $a_n$ are integers or not.

(20 points)

(8) Bonus Problem.

Let $K$ be a field, $F \in K[x_1, \ldots, x_n]$ a homogeneous polynomial of degree $d$. Show that $F(x_1, \ldots, x_n) = 0$ has a nontrivial solution in $K$ when $K = \mathbb{F}_p$ and $n > d$.

(20 points)

(9) Bonus Problem.

Let $F_0(x, y, z) = (x + y + z)xyz + x^2y^2 + y^2z^2 + z^2x^2 - x^4 - y^4 - z^4$,

$$F(x_1, \ldots, x_{18}) = \sum_{j=0}^{2} F_0(x_3j+1, x_3j+2, x_3j+3) + 4 \sum_{j=3}^{5} F_0(x_3j+1, x_3j+2, x_3j+3).$$

Show that $F(x_1, \ldots, x_{18}) = 0$ has no nontrivial solutions in $\mathbb{Q}_2$.

(10 points)