Introductory Number Theory - Problem Set 2
Spring 2008
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Due Friday, February 22, in class.

(1) Determine whether or not the following equations are solvable in integers:
   (a) \(2x^2 + 3 = y^2\)
   (b) \(2x^2 + 2 = 5y^2\)
   (c) \(x^3 + 10y^3 = 25z^3\) (where \((0, 0, 0)\) is not considered a solution)
   (d) \(x^5 + y^5 + z^5 = 4\) \(\quad (7+7+8+8 \text{ points})\)

(2) Find an integer \(x\) such that \(x \equiv 2 \text{ mod } 7\), \(x \equiv 1 \text{ mod } 11\) and \(x \equiv 9 \text{ mod } 13\).
   \(\quad (15 \text{ points})\)

(3) How many distinct solutions in \(\mathbb{Z}/1001\mathbb{Z}\) does the equation \(\bar{x}^2 = \bar{1}\) have?
   \text{HINT. } 1001 = 7 \cdot 11 \cdot 13. \quad (15 \text{ points})

(4) The largest known prime number is the Mersenne prime \(p = 2^{32,582,657} - 1\). Find
   its last six digits and its first six digits (how many decimal digits does it have?)
   without actually computing \(p\). You may (and probably should) use a computer
   for this.
   \text{HINT. } Logarithms have their uses. \quad (20 \text{ points})

(5) (a) Let \(p\) be a prime number, \(n \geq 0\). Show that
       \[v_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor\]
   (b) \textbf{BONUS PROBLEM}
   Let \(m, n \geq 0\) be integers. Show that \(\frac{(2m)! (2n)!}{m! n! (m + n)!}\) is an integer.
       \(\quad (20+10 \text{ points})\)

(6) \textbf{BONUS PROBLEM}
Fix the gap in the computation of the density of coprime pairs!
More precisely, show that the limit
\[P = \lim_{N \to \infty} \frac{\#\{(m, n) : 1 \leq m, n \leq N, m \perp n\}}{N^2}\]
exists. \(\quad (20 \text{ points})\)